GUARDING SIMPLE POLYGONS WITH SEMI-OPEN EDGE GUARDS

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ABSTRACT

In this note we explore an upper bound on the number of semi-open guard edges in a non star-shaped polygon and provide a characterization to detect semi-open guard edges. Based on this characterization we propose an $O(n)$ time algorithm to find all semi-open guard edges in a polygon.

KEYWORDS

Geometric Algorithms, Visibility, Art-Gallery Problems, Edge Guards

1 INTRODUCTION

The boundary structure of a simple polygon P can be very complex. Let $bd(P)$ denote its boundary. Visibility relationships between pairs of points of P have been proposed in the literature that shed more light on the structure of $bd(P)$. Two points p and q of P are visible to each other if segment $\overline{pq}$ lies entirely inside P. Thus if there exists a point inside P from which the entire boundary is visible then P is star-shaped.

This notion of visibility gave birth to an extensive literature on art-gallery problems [1], concerned with guarding the floor of a polgonal art-gallery. Chvátal [2] showed that $\lfloor n/3 \rfloor$ point guards are always sufficient and sometimes necessary to guard an art-gallery in the shape of a simple polygon. Allowing guards to move on an edge gives rise to the class of edge guard problems. This notion of visibility from an edge, or from a moving guard, also generated an extensive research literature. This is not surprising, since a polygon is a model of geographical space, and an edge guard is not only a model for visibility, but also for illumination with linear light sources, as well as communication capability between different parts of the space (polygon). Thus the work presented here has applications to computer graphics [3], to communication networks, as well as path planning in robotics with limited visibility [4], [5].

An edge guard is closed if the end-points of the edge are included, semi-open if one end-point is included, and open if both end-points are excluded. Shermer [6] established an upper bound of $\lfloor 3n/10 \rfloor + 1$ on the number of closed edge guards needed, and also showed that $\lfloor n/4 \rfloor$ guards are sometimes necessary.

In [7], Toth et al. has shown that a non star-shaped simple polygon can have at most one open guard edge. Park et al. [8] showed that such a polygon can have at most 3 closed guard edges. Thus it is interesting to explore the scenario in which the edge guards are semi-open. A semi-open edge guard includes exactly one of the end-points. For clarity and focus, in this paper the included end-point is always the end that is met first in a clockwise traversal of the polygon P. We show that a non star-shaped polygon has at most 3 semi-open guard edges and propose an algorithm to find all semi-open guard edges of a polygon.

The notational infrastructure is the same as that of [7]; we introduce these as we progress in the paper.

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2 SEMI-OPEN GUARD EDGES

Lemma 1 *Let $e = (u, v]$ be a semi-open edge of a polygon P and p a point interior to it. Then p is visible from e iff the set of common vertices of the paths $p \rightsquigarrow u$ and $p \rightsquigarrow v$ is either $\{p\}$ or $\{p, v\}$.*

Proof: If the set of common vertices of the paths $p \rightsquigarrow u$ and $p \rightsquigarrow v$ is $\{p\}$, then $\overline{uv}$, and the paths from p to u and v form a pseudo-triangle; as each corner of this triangle is visible from the opposite side, p is visible from $\overline{uv}$. When the set of common vertices is $\{p, v\}$, p is directly visible from v.

If the paths $p \rightsquigarrow u$ and $p \rightsquigarrow v$ share a common vertex $q \notin \{p, v\}$, then every geodesic path from p to a point on $\overline{uv}$ passes through q and hence p is not visible from $\overline{uv}$. □

The example of Figure 1 shows that a non star-shaped polygon can have 3 semi-open edge guards. This motivates the following theorem.

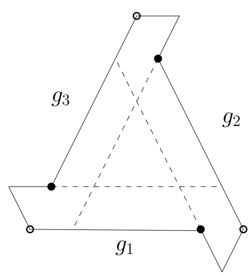

Figure 1: A non-starshaped polygon that has three semi-open guard edges : g_1, g_2 and g_3

Theorem 1 *Every non star-shaped simple polygon has at most three semi-open guard edges.*

Proof: Let g_1, g_2, g_3 and g_4 be 4 semi-open guard edges, positioned in counter-clockwise order around the polygon P.

Let $g_1 = (a, b]$ and $g_3 = (c, d]$. There are two cases to consider:

(i) The end points a, b, c and d are in convex position.

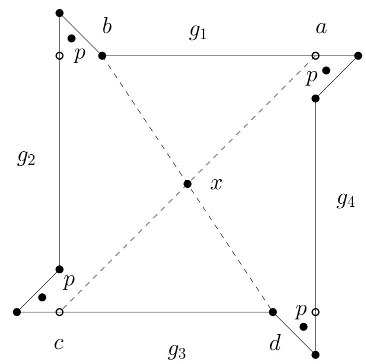

Figure 2: Structure of a polygon that has four semi-open guard edges

In this case, the structure of a polygon that has four semi-open guard edges is shown in Figure 2.

Since b is visible from the guard edge g_3, the geodesic paths from c and d to b have only b as a common vertex; again, since d is visible from g_1, only d is common to the geodesic paths from d to a and b. Thus the geodesic path from b to d separates the geodesic paths from b to c and a to d. Thus the path from b to d is a straight edge. By a similar argument, the path from a to c is a straight edge.

Consider p in the niche between the guard edges g_1 and g_2. Assume that it is not visible from x. The path $p \rightsquigarrow x$ makes a last left turn at b; this is also where the path $p \rightsquigarrow a$ makes a left turn, as p is visible from $[b, a)$. This means that p is not visible from any point on g_4, since any direct line of visibility must cross the segments $\overline{bd}$ and $\overline{ac}$. Thus p is visible from x, making P star-shaped.

(ii) The end points a, b, c and d are not in convex position, with $CH(\{a, b, c, d\}) = \triangle abc$.

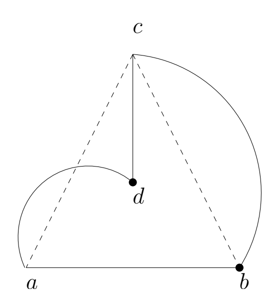

Figure 3: Structure of a polygon that has four semi-open guard edges when $CH(\{a, b, c, d\}) = \triangle abc$

It is clear from Figure. 3 that in this case it is impossible to put a semi-open guard edge on the part of $bd(P)$ that goes from d to a. Thus it is impossible to have 4 semi-open guard edges in this case.

3 CHARACTERIZING SEMI-OPEN GUARD EDGES

Let r be a reflex vertex of P. With respect to a counter-clockwise order of $bd(P)$, let r^- be the vertex that precedes r on $bd(P)$, and r^+ the one that succeeds it. Let p^- be the intersection with $bd(P)$ of a ray shot from r in the direction $\overrightarrow{r^-r}$, while p^+ is the intersection with a ray shot from r in the direction $\overrightarrow{r^+r}$.

These rays define two polygons: a left polygon $C_{left}(r)$ bounded by the chord rp^- and the part of $bd(P)$ from p^- to r in the counter-clockwise order; and a right polygon $C_{right}(r)$ bounded by the chord rp^+ and the part of $bd(P)$ from r to p^+ in the counter-clockwise order.

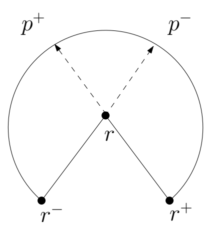

Figure 4: Two subpolygons defined by a reflex vertex r

The left (respectively, right) kernel, $K_{left}(P)$ (respectively, $K_{right}(P)$) is the intersection of all the left (respectively, right) polygons $C_{left}(r)$ (respectively, $C_{right}(r)$), while the kernel of P is the intersection of $K_{left}(P)$ and $K_{right}(P)$.

Fact 1 *The kernels $K_{left}(P)$ and $K_{right}(P)$ are both convex.*

Toth et al. [7] showed that the left and right kernels can be used to define left-kernel and right-kernel decompositions of $int(P)$. These decompositions were then used to prove the following theorem.

Theorem 2 *A open edge $e = (a,b)$ of a simple polygon P is an guard edge iff e intersects both the left and right kernels of P.*

The above theorem tacitly assumes that both kernels are non-empty. Now, it is possible that one of the kernels is empty; the above theorem does not address this situation. Does the theorem still hold? It certainly does not if we substitute open guard edge with semi-open guard edge. Consider the example in Figure. 5. The right kernel is empty, while the left kernel is not. The edges $(p_0, p_1]$ and $(p_3, p_4]$ both intersect the left kernel, and as the right kernel is empty the intersection of these edges with the right kernel is null. Nevertheless, $(p_0, p_1]$ is a semi-open edge guard, while $(p_3, p_4]$ is not. Figure. 6 shows a slightly more complex example in which $(p_7, p_8]$ is the only semi-open guard edge.

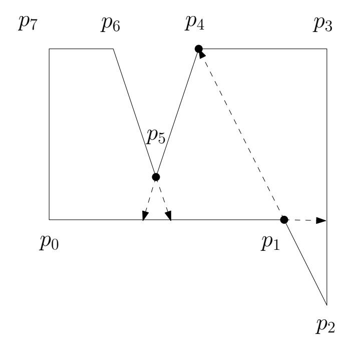

Figure 5: A left or right kernel can consist of several convex pieces

Thus we need to restate the above theorem to characterize a semi-open guard edge.

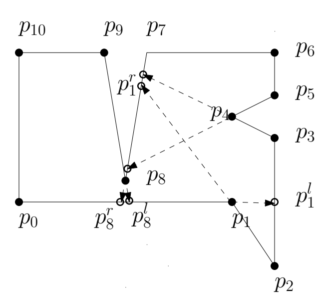

Figure 6: A left or right kernel can consist of several convex pieces

Theorem 3 *A semi-open edge $e = (a,b]$ is an guard edge iff e has a non-empty intersection with $C_{left}(r) \cap C_{right}(r)$ for every reflex vertex.*

Proof. Let $e = (a,b]$ be a semi-open guard edge and p be a point of the polygon which is not visible from e. Then the geodesic paths $p \rightsquigarrow a$ and $p \rightsquigarrow b$ have a common reflex vertex r that is not in $\{p,b\}$ (see Figure. 7).

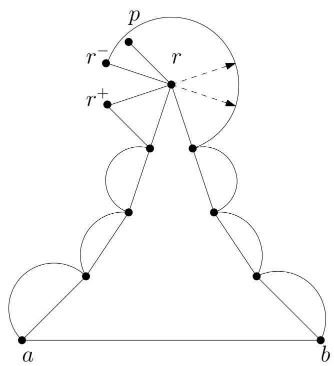

Figure 7: $C_{left}(r) \cap C_{right}(r)$ disjoint from e

Since a ray shot in the direction p to r, incident on $\overline{pr}$, does not hit $(a,b]$, and the edges of P incident on the reflex vertex r are situated counter-clockwise with respect to $\overline{pr}$, rays shot along these edges do not intersect $(a,b]$ either. Thus $C_{left}(r) \cap C_{right}(r)$ does not have a non-empty intersection with $(a,b]$; a contradiction.

Let r be a reflex vertex such that $C_{left}(r) \cap C_{right}(r)$ has an empty intersection with e. Also let e lie in $C_{left}(r)$. If p be a point interior to $\overline{r^{-}r}$, then the the geodesic paths from p to the end-points a and b of e share r as a common interior vertex. Thus p is not visible from e, which is not an guard edge; a contradiction. □

4 ALGORITHM

In [9], Bhattacharya et al. proposed a linear time algorithm for computing a shortest internal line segment l from which a polygon P is weakly internally visible. Central to their algorithm is the notion of a C-polygon. Both $C_{left}(r)$ and $C_{right}(r)$, as defined in this paper, are C-polygons of P. A C-polygon is non-redundant if it does not properly contain any other C-polygon. Thus referring to the example of Figure. 6, $C_{left}(p_1)$ is redundant since it contains the C-polygon $C_{right}(p_4)$). Indeed, in this example we have 3 non-redundant C-polygons and these are, in addition to $C_{right}(p_4)$, $C_{right}(p_1)$ and $C_{right}(p_8)$. They proved the following result.

Lemma 2 *P is weakly internally visible from a line segment $l = \overline{uv}$ iff l intersects every non-redundant C-polygon of P.*

They also showed how to compute all non-redundant C-polygons in linear time.

Since P is weakly visible from a semi-open guard edge, such an edge must intersect all non-redundant C-polygons. Thus we have an alternate characterization of a semi-open (in fact, even for open and closed) guard edge, that leads to a linear time algorithm to determine all semi-open guard edges.

Let the ends of the bounding chord of each non-redundant C-polygon that lie on $bd(P)$ be marked blue and red in counter-clockwise order in an initial counter-clockwise traversal of $bd(P)$.

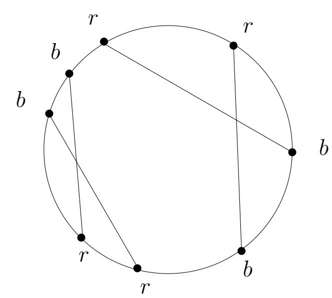

Figure 8: Colouring the end points of a bounding chord of a non-redundant C-polygon

Let the integer variable *edgeCount(e)* be a count of the number of non-redundant C-polygons that have a non-empty intersection with an edge e of P. An edge e is deemed to intersect a C-polygon in one of the 3 situations shown in Figure. 9.
For each edge e, $edgeCount(e)$ is set to the number of blue and red points contained in the semi-open edge e. We traverse $bd(P)$ once to do this.

In order to correct this count with respect to the number of C-polygons in which e is contained (rightmost configuration in Figure. 9), we traverse $bd(P)$ twice,

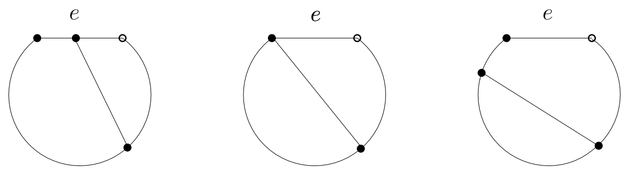

Figure 9: Counting intersections of an edge e with a non-redundant C-polygon

incrementally subtracting from *edgeCount(prev(e))* the number of red points in $prev(e)$ and adding the difference to *edgeCount(e)*. Two traversals are needed since the *edgeCount* can be an underestimate for the initial edge e_1, from where the traversal starts.

At termination of the second round, we declare those edges e as semi-open guard edges whose intersection count *edgeCount(e)* is equal to the number of non-redundant components.

Simulating the above algorithm on the example of Figure. 6, we find that the only semi-open edge whose *edgeCount(e)* is 3 is $e = (p_7, p_8]$.

5 POLYGONS WITH HOLES

A polygon P with holes can have guard edges that lie on the outer boundary or on the boundaries of the holes. Park et al. [8] have shown that to establish upper bounds it is enough to consider polygons with only one convex hole, indeed just one triangular hole. It is quite obvious that no semi-open edge of this triangular hole can be a guard edge as it cannot see all the points on its own boundary. As for guard edges on the outer boundary, the following theorem of [8] for closed guard edges carries over when the guard edges are semi-open.

Theorem 4 *For a polygon P with a convex hole H, the number of guard edges is at most 3.*

Thus a polygon P with holes can have at most 3 semi-open guard edges. An example is illustrated in Figure. 10.

6 1-HOLE ALGORITHM

Let P be a polygon with one triangular hole H. To find all semi-open guard edges of P, we start

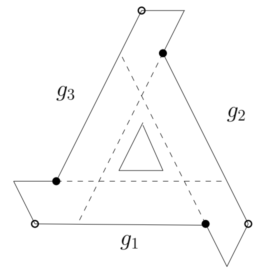

Figure 10: A polygon with a hole that has 3 semi-open guard edges

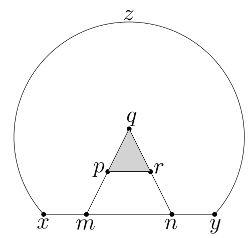

Figure 11: A polygon with a single triangular hole showing the cuts p to m and r to n

by finding at most 3 candidate edges and then narrow down from there. The candidate edges are the semi-open guard edges of P without H. These can be found by running our algorithm from Section 4.

In the next step, for each edge output by the first step we check if H is weakly externally visible from it. The set of edges from which H is weakly visible is input to the third and last step.

Let $e = (x, y]$ be a candidate edge from the second step. We attach H to P by inserting two cuts. The first cut is made by shooting a ray from q through p to intersect e at m, while the second cut is made by shooting a ray from q through r to intersect e at n. This gives us a new polygon $N =< q, p, m, x, \ldots, z, \ldots, y, n, r >$. We run our algorithm from Section 4 on N and determine if each non-redundant C-polygon intersects one of the edges xm, mp, rn, ny. If so, e is a semi-open guard edge, else not. We repeat this for the remaining candidate edges.

7 CONCLUSION

By considering semi-open edge guards, we are led to some interesting conclusions. The upper bound on the number of semi-open guard edges is the same as for closed edge guards. A more careful characterization is needed for a semi-open edge guard as one or both kernels can be empty.

It would also be interesting to find tight upper and lower bounds on the number of semi-open edge guards needed to guard a polygon P. The classes of polygons in Figure. 12 - Figure. 14 seem to suggest a lower bound of $2n/7$ semi-open edge guards.

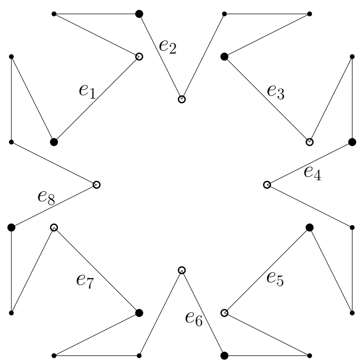

Figure 12: Semi open edges $e_1 - e_8$ guard this polygon

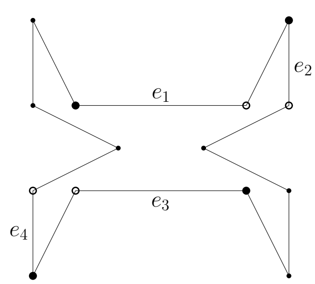

Figure 13: Semi open edges $e_1 - e_4$ guard this polygon

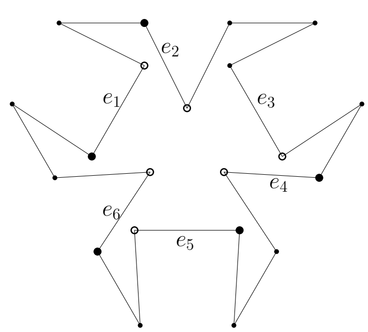

Figure 14: Semi open edges $e_1 - e_6$ guard this polygon

8 ACKNOWLEDGEMENTS

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