Fuzzy Malmquist Productivity Index

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ABSTRACT

The Malmquist productivity index (MPI) has been widely used in measuring the change in performance of a unit between two periods. This paper extends the conventional MPI from an environment of certainty to uncertainty, where the imprecise observations are represented by fuzzy numbers. An example explains how to derive the fuzzy MPI analytically from the fuzzy observations for very simple problems. For general problems, the extension principle is applied to formulate the problem of finding the α-cut of the fuzzy MPI as a pair of two-level mathematical programs. The two-level program is converted to a bound-constrained nonlinear program, and a quasi-Newton method is devised for a solution. The fuzzy MPI is derived numerically from the α-cuts at various α values. The case of Taiwanese forests after reorganization is used to illustrate the validity of the proposed model and the solution method devised for obtaining the fuzzy MPI. The results also demonstrate that the fuzzy approach provides more information than the conventional approach, which simplifies the problem by assuming the imprecise observations to have precise values.

KEYWORDS

Data envelopment analysis; Malmquist productivity index; Fuzzy numbers; Two-level mathematical programming; Nonlinear programming

1 INTRODUCTION

Efficiency is an effective measure of the performance of a production unit regarding the utilization of resources. When there are no absolute standards for calculating absolute efficiency, relative efficiency is usually measured by comparing a unit with other similar ones. Charnes et al. [8] developed the data envelopment analysis (DEA) technique to calculate the relative efficiency of a set of decision making units (DMUs) which apply the same multiple inputs to produce the same multiple outputs in a period of time, and this technique has seen wide application in the real world (see, for example, the review of Cook and Seiford [10]).

The DEA efficiency of a DMU can be calculated for different periods of time. Since it is a relative measure, as compared to other DMUs, in a specific period, it is possible that a DMU is efficient in two periods while its performance is actually declining. It is also possible that the efficiency scores of a DMU calculated from two different periods are decreasing when its performance is actually improving. The reason is simply because the relative efficiencies of the two periods are not calculated on a common basis. To make the efficiencies calculated from different periods comparable, a common basis is required. To this end, Caves et al. [3,4] proposed the Malmquist productivity index (MPI), which is the ratio of the efficiency of the next period to that of the current one, based on the production technology of the current period. A value greater than 1 indicates that the efficiency has improved between the two periods; otherwise, it has worsened.

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Conceptually, any period, not only the current one, can be used as the base period for calculating efficiency. Different base periods, however, may produce different MPIs. To solve this problem, Färe et al. [15] suggested using both current and next periods to calculate two MPIs separately, and taking their geometric average as the final MPI. Pastor and Lovell [29] used a combined production technology of the current and next periods to calculate the efficiencies at the two periods to produce a global MPI.

The MPI has been widely applied to measuring performance changes, especially due to an act or policy [1,6,19]. The conventional MPIs are calculated in environments of certainty where the observations can be measured precisely. In the real world, however, this is not always the case. Sometimes the observations are missing, and have to be estimated. Sometimes the event has not occurred yet, and the observations have to be predicted. There are also cases in which the data is qualitative, such as the verbal expressions of *excellent*, *good*, *fair*, *unsatisfactory*, and *poor*, which cannot be quantified precisely. In other words, there are uncertain situations to deal with, and uncertain observations are usually expressed as probabilistic, interval, and fuzzy measures.

In the DEA literature, Chambers [5], Kao and Liu [25], and Udhayakumar et al. [31] assumed the data to be probabilistic, described by specific probability distributions, when calculating efficiency. Cooper et al. [11,12], Despotis and Smirlis [13], Kao [20], and Park [28] represented the uncertain data by intervals. In contrast, Chin et al. [9], Guo [17], Guo et al. [18], Kao and Lin [23], Kao and Liu [24,26], and Lertworasirkul et al. [27] used fuzzy numbers. Since the MPI is the ratio of two efficiencies, a straightforward idea is, for each case, to find the ratio of two corresponding efficiencies, i.e., probabilistic, interval, and fuzzy. However, because the two efficiencies in the ratio are not independent, they cannot be calculated separately. Instead, they must be calculated for the same input-output instance, and then to find the distribution, interval, and membership function of the ratio of all instances for the three cases. This process is much more complicated than expected.

For fuzzy cases, Emrouznejad et al. [14] proposed a method for calculating the MPI. The idea is to apply a fuzzy ranking method to defuzzify the fuzzy models into crisp ones. The calculated MPI is a crisp number. Intuitively, when the observations are fuzzy numbers, the calculated MPI should also be fuzzy numbers. In this paper, we will develop a methodology to obtain the fuzzy MPI when the observations are represented by fuzzy numbers. The basic concept is to express the membership function of the fuzzy MPI as a function of those of the fuzzy observations based on the extension principle [32-34]. By viewing the membership functions from the α-cut, the idea of Kao and Liu [24] and Kao [22] is used to formulate the problem as a two-level mathematical program. The two-level program can be transformed into a nonlinear program where the objective function is not explicitly known. A modified quasi-Newton method is devised to solve the nonlinear problem. By enumerating various α values of the α-cuts, the membership function of the fuzzy MPI is obtained numerically.

This paper is organized as follows. In the next section, the concept of MPI is briefly reviewed. Then, in Section 3, a simple example is used to explain the characteristics of the fuzzy MPI, and an analytical solution method is introduced. Following that, a two-level mathematical programming model is developed in Section 4, based on the extension principle. To illustrate the methodology of this paper, the MPIs of the eight districts of Taiwanese forests after their reorganization in 1989 are calculated and discussed in Section 5. Finally, in Section 6, the conclusions of this work are presented.

2 THE MALMQUIST PRODUCTIVITY INDEX

Denote $X_{ij}^{(p)}$ and $Y_{rj}^{(p)}$ as the *i*th input, *i*=1,…,

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m, and rth output, r=1,…, s, respectively, of the jth DMU, j=1,…, n, in period p. The relative efficiency of DMU k under the assumption of variable returns to scale can be calculated via the following linear program [2]:

$$\begin{aligned} E_k^{(p)}(\boldsymbol{X},\boldsymbol{Y}) &= \max.\ \sum_{r=1}^{s} u_r Y_{rk}^{(p)} \\ \text{s.t. } & v_0 + \sum_{i=1}^{m} v_i X_{ik}^{(p)} = 1 \\ \sum_{r=1}^{s} u_r Y_{rj}^{(p)} &- (v_0 + \sum_{i=1}^{m} v_i X_{ij}^{(p)}) \le 0,\ j=1,\dots,n \\ & u_r, v_i \ge \varepsilon,\ r=1,\dots,s,\ i=1,\dots,m \\ & v_0 \text{ unrestricted in sign.} \end{aligned} \tag{1}$$

where ε is a small positive number imposed on the multipliers u_r and v_i to avoid ignoring any input/output factors in the evaluation [7]. This model finds the most favorable set of multipliers u_r and v_i for DMU k to yield the largest efficiency score under the condition that the same multipliers should not yield an efficiency score of greater than 1 for all DMUs. A value of 1 for $E_k^{(p)}$ indicates that DMU k is efficient in period p; otherwise, it is inefficient.
Caves et al. [3,4] calculated the MPI by calculating the relative efficiency of DMU k in period t+1 based on the technology of period t via the following linear program:

$$\begin{aligned} G_k^{(t)}(\boldsymbol{X},\boldsymbol{Y}) &= \max.\ \sum_{r=1}^{s} u_r Y_{rk}^{(t+1)} \\ \text{s.t. } & v_0 + \sum_{i=1}^{m} v_i X_{ik}^{(t+1)} = 1 \\ \sum_{r=1}^{s} u_r Y_{rj}^{(t)} &- (v_0 + \sum_{i=1}^{m} v_i X_{ij}^{(t)}) \le 0,\ j=1,\dots,n \\ & u_r, v_i \ge \varepsilon,\ r=1,\dots,s,\ i=1,\dots,m \\ & v_0 \text{ unrestricted in sign.} \end{aligned} \tag{2}$$

Sueyoshi [30] defined $G_k^{(t)}$ as the *growth index* of DMU k. The MPI is the ratio of $G_k^{(t)}$ to $E_k^{(t)}$. While Cave et al.'s MPI uses period t as the base period, period t+1 can also be used. In this case, the relative efficiency of DMU k in period t+1, $E_k^{(t+1)}$, is calculated via Model (1). The relative efficiency of DMU k in period t based on the technology of period t+1, $G_k^{(t+1)}$, is calculated as:

$$\begin{aligned} G_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y}) &= \max.\ \sum_{r=1}^{s} u_r Y_{rk}^{(t)} \\ \text{s.t. } & v_0 + \sum_{i=1}^{m} v_i X_{ik}^{(t)} = 1 \\ \sum_{r=1}^{s} u_r Y_{rj}^{(t+1)} &- (v_0 + \sum_{i=1}^{m} v_i X_{ij}^{(t+1)}) \le 0,\ j=1,\dots,n \\ & u_r, v_i \ge \varepsilon,\ r=1,\dots,s,\ i=1,\dots,m \\ & v_0 \text{ unrestricted in sign.} \end{aligned} \tag{3}$$

The relative efficiency $G_k^{(t+1)}$ can be interpreted as a growth index in a reverse sense; that is, growing from period t+1 to period t. The ratio of $E_k^{(t+1)}$ to $G_k^{(t+1)}$ is another MPI, based on the technology of period t+1.
The MPI based on period t+1 is usually different from that based on period t. Sometimes they may even yield inconsistent results, with one indicating an improvement while the other a worsening situation. To take the effect caused by different base periods into account, Färe et al. [15] suggested using the geometric mean of these two ratios as the MPI. In symbols, this is:

$$M_k(\boldsymbol{X},\boldsymbol{Y}) = \left[\frac{G_k^{(t)}(\boldsymbol{X},\boldsymbol{Y})}{E_k^{(t)}(\boldsymbol{X},\boldsymbol{Y})} \times \frac{E_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y})}{G_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y})} \right]^{1/2} \tag{4}$$

In this paper Equation (4) is used to calculate the MPI. However, the method developed here is applicable to other forms of MPI.

3 ANALYTICAL SOLUTION

When some $X_{ij}^{(p)}$ and $Y_{rj}^{(p)}$ observations are fuzzy numbers, it is expected that the MPI will also be a fuzzy number.

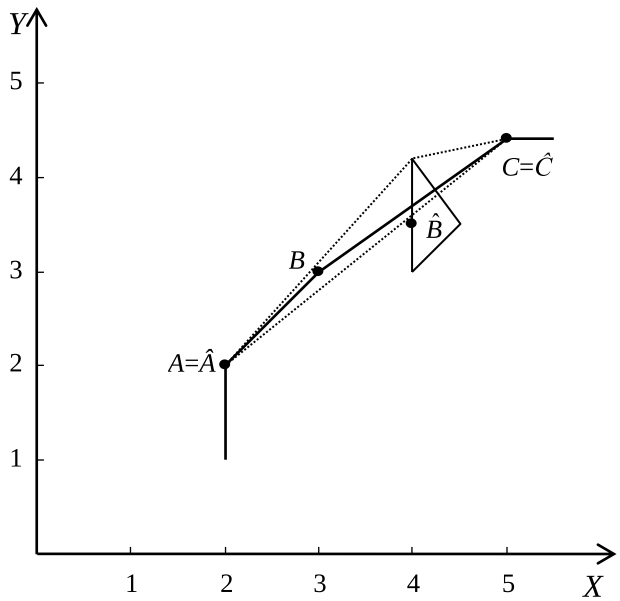

Figure 1. Graphical explanation for the calculation of MPI.

Consider a very simple case of three DMUs, *A*, *B*, and *C*, each uses one input *X* to produce one output *Y*. As depicted in Figure 1, *A*, *B*, and *C* use 2, 3, and 5 units of *X* to produce 2, 3, and 4.4 units of *Y*, respectively, in period *t*. In period *t*+1, *A* and *C* remain at the same position while *B* moves to $\hat{B}$, where 4 units of *X* are consumed to produce 3.5 units of *Y*. The production frontier constructed from *A*, *B*, and *C* in period *t* is the piecewise line segments *ABC*, which indicates that all three DMUs are efficient. The efficiency of $\hat{B}$ based on the production frontier *ABC* is the ratio of the output of $\hat{B}$, 3.5, to the projection of $\hat{B}$ on *ABC*, 3.7, which is 35/37. The MPI of DMU *B*, based on base year *t*, is thus (35/37)/1=0.9459. This value is smaller than 1, indicating that the performance of DMU *B* worsens from period *t* to period *t*+1. The efficiencies of $\hat{A}$ and $\hat{C}$, based on the production frontier *ABC*, are 1. Hence, they both have an MPI of 1, a sign of unchanged performance between the two periods.

The production frontier constructed from $\hat{A}$, $\hat{B}$, and $\hat{C}$ in period *t*+1 is the line segment $\hat{A}\hat{C}$, where DMUs *A* and *C* are efficient, while *B* is not. The efficiency score of DMU *B* is the ratio of the output of $\hat{B}$ to its projection on $\hat{A}\hat{C}$, which is 3.5/[(2+2 × 4.4)/3]=35/36. The efficiency of DMU *B* in period *t* based on the production frontier of period *t*+1, $\hat{A}\hat{C}$, is the ratio of the output of DMU *B* in period *t* to its projection on $\hat{A}\hat{C}$, which is 3/[2 × 2+4.4]/3]= 30/28. The MPI of DMU *B* based on period *t*+1 then is (35/36)/(30/28)=49/54=0.9074. Consequently, Färe's MPI for DMU *B*, according to Equation (4), is $[(35/37) \times (49/54)]^{0.5}$=0.9265. This value is smaller than Cave et al.'s MPI (which is 0.9459), and is smaller than 1. We are thus confident that the relative efficiency of DMU *B* is worsened.

The above discussion is for cases of certainty, but suppose the output of $\hat{B}$ is missing, and is estimated to be a triangular fuzzy number of $\tilde{Y}_{\hat{B}}$ = (3, 3.5, 4.2). In other words, $\tilde{Y}_{\hat{B}}$ has a membership function of:

$$\mu_{\tilde{Y}_{\hat{B}}}(y) = \begin{cases} (y-3)/0.5 & 3 \le y \le 3.5 \\ (4.2-y)/0.7 & 3.5 \le y \le 4.2 \end{cases} \quad (5)$$

as depicted by the triangle in Figure 1 on *X*=4. To calculate the MPI based on period *t*, note that the production frontier of period *t* is the piecewise line segments *ABC*, where $E_B^{(t)}$ =1. In period *t*+1, for *y* in the whole range of (3, 4.2), the growth index of DMU *B* is *y*/3.7, where 3.7 is the projection of *y* on the frontier *ABC*. Thus, the MPI of DMU *B*, based on the technology of period *t*, is (*y*/3.7)/1=*y*/3.7.

To calculate the MPI based on the technology of period *t*+1, the production frontier has two forms to discuss, for *y* in the ranges of [3, 3.6] and [3.6, 4.2].

(i) $y \in [3, 3.6]$

For *y* in the range of [3, 3.6], the production frontier constructed from $\hat{A}$, $\hat{B}$, and $\hat{C}$ is the line segment $\hat{A}\hat{C}$. In this case, the efficiency of $\hat{B}$ is *y*/3.6, and the reversal growth index for DMU *B*, as discussed in the deterministic case, is 30/28, which result in a value of (y/3.6)/(30/28)=7y/27 for the MPI based on period t+1. Combined with the MPI based on period t, which is y/3.7, the Färe's MPI for DMU B is M_B =[(y/3.7)(7y/27)]$^{0.5}$ = $\sqrt{70/999}\, y$ =0.2647y.

(ii) $y \in [3.6, 4.2]$

For y in the range of [3.6, 4.2], the production frontier constructed from $\hat{A}$, $\hat{B}$, and $\hat{C}$ is the kinked line $\hat{A}y\hat{C}$, where $\hat{B}$ is efficient with $E_{\hat{B}}$=1. The reversal growth index for DMU B is the output of B divided by its projection on $\hat{A}y\hat{C}$, which is 3/[(2+y)/2]=6/(2+y). The MPI of DMU B based on period t+1 is thus equal to 1/[6/(2+y)]=(2+y)/6. Combined with the MPI of period t, the Färe's MPI for DMU B is M_B= {(y/3.7)[(2+y)/6]}$^{0.5}$ =[5y(2+y)/111]$^{0.5}$.

In Case (i), the inverse function of MPI is $y=M/\sqrt{70/999}=\sqrt{999/70}\,M$. For y in the range of [3, 3.5], the corresponding range of MPI is [$\sqrt{70/111}$, $\sqrt{1715/1998}$], or [0.7941, 0.9265], its membership function, from Equation (5), is $\mu_{\tilde{Y}_{\hat{B}}}(y)$ = (y–3)/0.5. Substituting $\sqrt{999/70}\,M$ for y, one obtains the membership function for MPI as: $\mu_{\tilde{M}}(m)=\sqrt{1998/35}\,m$–6. For y in the range of [3.5, 3.6], the corresponding range of MPI is [$\sqrt{1715/1998}$, $\sqrt{1008/1110}$], or [0.9265, 0.9529], its membership function is $\mu_{\tilde{Y}_{\hat{B}}}(y)$ =(4.2–y)/0.7. Substituting $\sqrt{999/70}\,M$ for y, the membership function obtained for MPI becomes: $\mu_{\tilde{M}}(m)$ =6– $\sqrt{9990/343}\,m$= 6–5.3968m.

In Case (ii), the inverse function of MPI is $y=-1+\sqrt{1+22.2M^2}$. For y in the range of [3.6, 4.2], the corresponding range of MPI is [$\sqrt{1008/1110}$, $\sqrt{434/370}$], or [0.9529, 1.0830]. The membership function for $\tilde{Y}_{\hat{B}}$ is described by $\mu_{\tilde{Y}_{\hat{B}}}(y)$ =(4.2–y)/ 0.7. Substituting $y=-1+\sqrt{1+22.2M^2}$ into $\mu_{\tilde{Y}_{\hat{B}}}(y)$, one obtains the membership function of $\tilde{M}$ as: $\mu_{\tilde{M}}(m)$ = 6–(–1+ $\sqrt{1+22.2m^2}$)/0.7=5.2– $\sqrt{1+22.2m^2}$ /0.7 .

Combining these results together, the membership function for the fuzzy MPI is obtained as:

$$\mu_{\tilde{M}}(m)=\begin{cases}\sqrt{1998/35}\,m-6, & \sqrt{70/111}\le m\le\sqrt{1715/1998}\\ 6-\sqrt{9990/343}\,m, & \sqrt{1715/1998}\le m\le\sqrt{1008/1110}\\ (5.2-\sqrt{1+22.2m^2}/0.7, & \sqrt{1008/1110}\le m\le\sqrt{434/370}\end{cases}$$

Note that the term $\sqrt{1+22.2m^2}$ in the third part of $\mu_{\tilde{M}}(m)$ is a nonlinear function of m, although it is quite close to the linear function of $\sqrt{23.2}\,m$. Figure 2 shows the shape of this membership function.

This example shows that when some observations are fuzzy numbers, the MPI is also a fuzzy number. For very simple problems, the membership function of the MPI can be derived analytically.

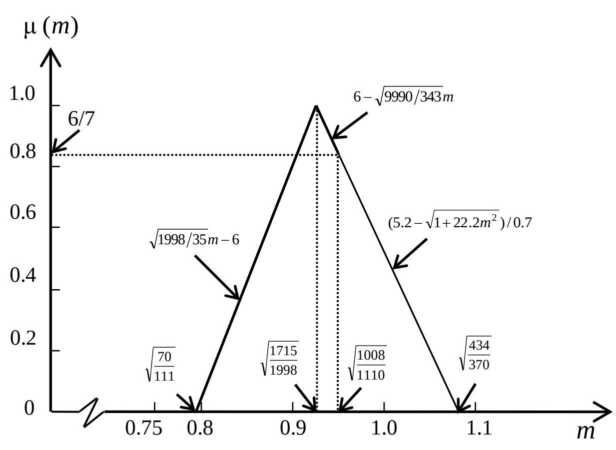

Figure 2. Fuzzy MPI of DMU B of the analytical example.

4 NUMERICAL SOLUTION

The example in the preceding section shows that the analytical derivation of the membership function of the fuzzy MPI from those of the fuzzy observations is very complicated, even for a simple problem of three DMUs with only one fuzzy observation. The problem becomes intractable when the number of DMUs or fuzzy observations is large. In this section, a methodology for numerically obtaining the fuzzy MPI for general problems is developed.

Without loss of generality and for convenience of notation, assume that all observations are fuzzy numbers, as crisp values can be considered as degenerated fuzzy numbers with only one point in the domain. Let $\tilde{X}_{ij}^{(p)}$ and $\tilde{Y}_{rj}^{(p)}$ denote the fuzzy counterpart of $X_{ij}^{(p)}$ and $Y_{rj}^{(p)}$, and denote $\mu_{\tilde{X}_{ij}^{(p)}}$ and $\mu_{\tilde{Y}_{rj}^{(p)}}$ as the membership functions of $\tilde{X}_{ij}^{(p)}$ and $\tilde{Y}_{rj}^{(p)}$, respectively. When observations are fuzzy numbers, the measured efficiency, growth index, and MPI will also be fuzzy numbers. Based on the extension principle [32-34], the membership function of the fuzzy MPI of DMU k, $\mu_{\tilde{M}_k}$, satisfies the following equation:

$$\mu_{\tilde{M}_k}(m) = \sup_{\boldsymbol{x},\boldsymbol{y}} \min\{ \mu_{\tilde{X}_{ij}^{(p)}}(x_{ij}^{(p)}),\ \mu_{\tilde{Y}_{rj}^{(p)}}(y_{rj}^{(p)}),\ \forall\, i, r, p, j \mid m = M_k(\boldsymbol{x},\boldsymbol{y})\} \qquad (6)$$

where M_k is defined in Equation (4).

Equation (6) shows the relationship between the membership function of the fuzzy M_k and those of the fuzzy observations. Using this expression directly to calculate $\mu_{\tilde{M}_k}$ requires solving a mathematical program that involves another mathematical program, i.e. $m=M_k(\boldsymbol{x},\boldsymbol{y})$ of Equation (4), as a constraint, which is intractable. Equation (6) views the membership function vertically, looking at the membership grade of $\tilde{M}_k$ for each value in its domain. Another approach is to view the problem horizontally, looking at the α-cut of $\tilde{M}_k$.

Denote $(X_{ij}^{(p)})_\alpha = [\,(X_{ij}^{(p)})_\alpha^L,\ (X_{ij}^{(p)})_\alpha^U\,]$ and $(Y_{rj}^{(p)})_\alpha = [\,(Y_{rj}^{(p)})_\alpha^L,\ (Y_{rj}^{(p)})_\alpha^U\,]$ as the α-cuts of $\tilde{X}_{ij}^{(p)}$ and $\tilde{Y}_{rj}^{(p)}$, respectively. In Expression (6), $\mu_{\tilde{M}_k}(m)$ is essentially the minimum of $\mu_{\tilde{X}_{ij}^{(p)}}(x_{ij}^{(p)})$ and $\mu_{\tilde{Y}_{rj}^{(p)}}(y_{rj}^{(p)})$, $\forall\ i, r, p, j$. Suppose $\mu_{\tilde{M}_k}(m)$ has a value of α. To satisfy $\mu_{\tilde{M}_k}(m) = \alpha$, it is necessary that $\mu_{\tilde{X}_{ij}^{(p)}}(x_{ij}^{(p)}) \ge \alpha$ and $\mu_{\tilde{Y}_{rj}^{(p)}}(y_{rj}^{(p)}) \ge \alpha$, and at least one of $\mu_{\tilde{X}_{ij}^{(p)}}(x_{ij}^{(p)})$ and $\mu_{\tilde{Y}_{rj}^{(p)}}(y_{rj}^{(p)})$, $\forall\ i, r, p, j$, is equal to α, such that $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ produce an MPI score equal to m in Model (4).

All α-cuts form a nested structure with respect to α [34]; that is, given $0<\alpha_2<\alpha_1 \le 1$, one has $[\,(X_{ij}^{(p)})_{\alpha_1}^L,\ (X_{ij}^{(p)})_{\alpha_1}^U\,] \subseteq [\,(X_{ij}^{(p)})_{\alpha_2}^L,\ (X_{ij}^{(p)})_{\alpha_2}^U\,]$. Hence, $\mu_{\tilde{X}_{ij}^{(p)}}(x_{ij}^{(p)}) \ge \alpha$ and $\mu_{\tilde{X}_{ij}^{(p)}}(x_{ij}^{(p)}) = \alpha$ correspond to the same α-cut. This is also true for $\mu_{\tilde{Y}_{rj}^{(p)}}(y_{rj}^{(p)})$. To find the membership function $\mu_{\tilde{M}_k}(m)$, it suffices to find the lower and upper bounds of the α-cut of $\tilde{M}_k$, $(M_k)_\alpha = [\,(M_k)_\alpha^L, (M_k)_\alpha^U\,]$. The upper bound $(M_k)_\alpha^U$ is equal to $\max\{m \mid \mu_{\tilde{M}_k}(m) \ge \alpha\}$, and the lower bound $(M_k)_\alpha^L$ is equal to $\min\{m \mid \mu_{\tilde{M}_k}(m) \ge \alpha\}$, where m is defined in Equation (4). Thus, $(M_k)_\alpha^L$ and $(M_k)_\alpha^U$ can be calculated via the following two-level mathematical programs:

$$(M_k)_\alpha^L = \min_{\substack{(X_{ij}^{(p)})_\alpha^L \le x_{ij}^{(p)} \le (X_{ij}^{(p)})_\alpha^U \\ (Y_{rj}^{(p)})_\alpha^L \le y_{rj}^{(p)} \le (Y_{rj}^{(p)})_\alpha^U \\ \forall\, i,r,p,j}} \left[\frac{G_k^{(t)}(\boldsymbol{X},\boldsymbol{Y})}{E_k^{(t)}(\boldsymbol{X},\boldsymbol{Y})} \times \frac{E_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y})}{G_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y})} \right]^{1/2} \qquad (7a)$$

$$(M_k)_\alpha^U = \max_{\substack{(X_{ij}^{(p)})_\alpha^L \le x_{ij}^{(p)} \le (X_{ij}^{(p)})_\alpha^U \\ (Y_{rj}^{(p)})_\alpha^L \le y_{rj}^{(p)} \le (Y_{rj}^{(p)})_\alpha^U \\ \forall\, i,r,p,j}} \left[\frac{G_k^{(t)}(\boldsymbol{X},\boldsymbol{Y})}{E_k^{(t)}(\boldsymbol{X},\boldsymbol{Y})} \times \frac{E_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y})}{G_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y})} \right]^{1/2} \qquad (7b)$$

For each set of $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ values selected from the respective α-cuts in the outer program (first level), the MPI is calculated in the inner program (second level), which involves four linear programs. The sets of $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ values which produce the smallest and largest MPI values are determined at the first level. Since the feasible region for variables $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ is a hyper-rectangle, which is a convex and compact set, the bounds $(M_k)_\alpha^L$ and $(M_k)_\alpha^U$ are continuous with respect to $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$. Based on this concept, a numerical method can be devised to solve this pair of two-level programs.

The basic idea is to treat the second-level program as a function of $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$, $f(\boldsymbol{x}, \boldsymbol{y})=[(G_k^{(t)}(\boldsymbol{X},\boldsymbol{Y}) \times E_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y}))/(E_k^{(t)}(\boldsymbol{X},\boldsymbol{Y}) \times G_k^{(t+1)}(\boldsymbol{X},\boldsymbol{Y}))]^{1/2}$, although the function form of $f(\boldsymbol{x}, \boldsymbol{y})$ is not explicitly known. In this case, Models (7a) and (7b) become unconstrained nonlinear programs with bound constraints. Starting with a set of $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ values in their respective α-cuts, a descending direction (or ascending direction for Model (7b)) is generated to find a new set of $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ values in their respective α-cuts along this direction with a better value of $f(\boldsymbol{x}, \boldsymbol{y})$ (smaller for (7a) and larger for (7b)). With this new set of $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ values, the process starts anew. This process is repeated until no better set of $x_{ij}^{(p)}$ and $y_{rj}^{(p)}$ values can be found. The lower bound $(M_k)_\alpha^L$ and upper bound $(M_k)_\alpha^U$ are then found. Note that to calculate $f(\boldsymbol{x}, \boldsymbol{y})$, four linear programs need to be solved.

In this paper, the quasi-Newton method, with a BFGS formula [16], is used to find the optimal solution of $f(\boldsymbol{x}, \boldsymbol{y})$. Different from the unconstrained problem, every time when the boundary of the feasible region is reached in the line search, the updating of the BFGS formula starts anew. The basic algorithm is as follows.

Step 1 Initialization

1.1 Set ($\boldsymbol{x}^{(0)}$, $\boldsymbol{y}^{(0)}$) to the center point, (($\boldsymbol{X}_\alpha^L + \boldsymbol{X}_\alpha^U$)/2, ($\boldsymbol{Y}_\alpha^L + \boldsymbol{Y}_\alpha^U$)/2), step=$c$, n=0.

1.2 Calculate $\nabla f(\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)})$, approximated by the central difference.

1.3 Set $\boldsymbol{H}^{(n)}=\boldsymbol{I}$, $\boldsymbol{d}^{(n)} = \nabla f(\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)})$ ($\boldsymbol{d}^{(n)} = -\boldsymbol{d}^{(n)}$ for minimization).

Step 2 Termination check

If $\| \boldsymbol{d}^{(n)} \| < \varepsilon$, then terminate, with MPI= $f(\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)})$; otherwise, continue Step 3.

Step 3 Line search

3.1 Set ($\hat{\boldsymbol{x}}^{(0)}, \hat{\boldsymbol{y}}^{(0)}$)=($\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)}$), k=0.

3.2 Set ($\hat{\boldsymbol{x}}^{(k+1)}, \hat{\boldsymbol{y}}^{(k+1)}$)=($\hat{\boldsymbol{x}}^{(k)}, \hat{\boldsymbol{y}}^{(k)}$)+step$\times \boldsymbol{d}^{(n)}$

3.3 Boundary check

If $\hat{x}_{ij}^{(p)(k+1)} < (X_{ij}^{(p)})_\alpha^L$, then set $\hat{x}_{ij}^{(p)(k+1)} = (X_{ij}^{(p)})_\alpha^L$, $\forall\, i, j, p$

If $\hat{x}_{ij}^{(p)(k+1)} > (X_{ij}^{(p)})_\alpha^U$, then set $\hat{x}_{ij}^{(p)(k+1)} = (X_{ij}^{(p)})_\alpha^U$, $\forall\, i, j, p$

If $\hat{y}_{rj}^{(p)(k+1)} < (Y_{rj}^{(p)})_\alpha^L$, then set $\hat{y}_{rj}^{(p)(k+1)} = (Y_{rj}^{(p)})_\alpha^L$, $\forall\, r, j, p$

If $\hat{y}_{rj}^{(p)(k+1)} > (Y_{rj}^{(p)})_\alpha^U$, then set $\hat{y}_{rj}^{(p)(k+1)} = (Y_{rj}^{(p)})_\alpha^U$, $\forall\, r, j, p$

If any of the above conditions occur, then set ($\boldsymbol{x}^{(n+1)}, \boldsymbol{y}^{(n+1)}$)=($\hat{\boldsymbol{x}}^{(k+1)}, \hat{\boldsymbol{y}}^{(k+1)}$), n=n+1, and go to Step 1.2; otherwise, calculate $f(\hat{\boldsymbol{x}}^{(k+1)}, \hat{\boldsymbol{y}}^{(k+1)})$ and continue Step 3.4.

3.4 If $f(\hat{\boldsymbol{x}}^{(k+1)}, \hat{\boldsymbol{y}}^{(k+1)})>f(\hat{\boldsymbol{x}}^{(k)}, \hat{\boldsymbol{y}}^{(k)})$ (replace ">" with "<" for minimization), then set step=2 × step, k=k+1 and go to Step 3.2; otherwise, set ($\boldsymbol{x}^{(n+1)}, \boldsymbol{y}^{(n+1)}$)=($\hat{\boldsymbol{x}}^{(k)}, \hat{\boldsymbol{y}}^{(k)}$) and continue Step 4.

Step 4 Direction generation

4.1 Calculate $\Delta g = \nabla f(\boldsymbol{x}^{(n+1)}, \boldsymbol{y}^{(n+1)}) - \nabla f(\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)})$, Δz=($\boldsymbol{x}^{(n+1)}, \boldsymbol{y}^{(n+1)}$)– ($\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)}$), $\boldsymbol{H}^{(n+1)} = \left[\boldsymbol{I} - \frac{\Delta z^t \Delta g}{\Delta z \Delta g^t}\right] \boldsymbol{H}^{(n)} \left[\boldsymbol{I} - \frac{\Delta z^t \Delta g}{\Delta z \Delta g^t}\right] + \frac{\Delta z^t \Delta g}{\Delta z \Delta g^t}$.

4.2 Calculate $\boldsymbol{d}^{(n+1)} = \nabla f(\boldsymbol{x}^{(n+1)}, \boldsymbol{y}^{(n+1)})\, \boldsymbol{H}^{(n+1)}$ ($\boldsymbol{d}^{(n+1)} = -\boldsymbol{d}^{(n+1)}$ for minimization).

4.3 Set n=n+1 and go to Step 2.

Note that the line search in Step 3 is inexact. However, since the BFGS method is robust for inexact line search, the inexactness of Step 3 does not affect the convergence of the algorithm. An attractive point of this method is that Model (7) is essentially an unconstrained nonlinear program; therefore, problems of hundreds of variables are fairly easy to solve.

In the next section, the changes in performance of Taiwanese forests after their reorganization in 1989 is used to illustrate the calculation of fuzzy MPI when some observations are fuzzy numbers, as discussed in this section.

5 CHANGES IN PERFORMANCE OF TAIWANESE FORESTS AFTER REORGANIZATION

The national forests in Taiwan were originally divided into thirteen districts by the Taiwan Forestry Bureau (TFB). As the districts were relatively small in area, the TFB reorganized them into eight larger ones in 1989, hoping that the management could be more efficient. After three years, a follow-up evaluation was conducted in 1992 to investigate whether the efficiency had improved. Kao [19] presented a detailed description of the follow-up evaluation, and Kao [21] used a common-weight MPI to measure the changes in efficiency after the reorganization. The imprecise data were assumed to be precise in these two periods.

Four inputs and three outputs are considered in Kao [19,21]. The four inputs are:

Land (X_1): area in hectares.

Labor (X_2): number of employees.

Expenditure (X_3): expenses per year in US dollars.

Initial stock (X_4): volume of forest stock before the evaluation in cubic meters.

The three outputs are:

Timber production (Y_1): timber harvested each year in cubic meters.

Soil conservation (Y_2): forest stock for conserving soil in cubic meters.

Recreation (Y_3): visitors served by forests every year in number of visits.

To measure the changes in efficiency of the eight new districts, thirteen old districts and four similar forests were added in Kao [19,21] to increase the number of DMUs so that a more reliable result was assured. That is, the seventeen old forests and the eight new districts in 1989 were used to construct the production frontier for period t and the seventeen old forests and eight new districts in 1992 were used to construct the production frontier for period t+1. Based on the two frontiers, the MPIs of the eight new districts between the two periods were calculated. Table 1 shows the data of the seventeen old forests and Table 2 shows the data of the eight new districts before and after the reorganization.

Of the seven input-output factors, the forest stocks (X_4 and Y_2) are unable to be measured precisely. Suppose they can be represented by isosceles triangular fuzzy numbers with the most likely numbers, shown in Tables 1 and 2, as the vertex of the triangle and $\pm 0.5\%$ of that value as the support. For example, the initial stock of Wen Shan District, in the first row of Table 1, is 5.04 million cubic meters. Therefore, its corresponding support is $(1 \pm 0.5\%) \times 5.04$= [5.01, 5.07], and the associated fuzzy number is $\tilde{X}_{41}$ = (5.01, 5.04, 5.07). Its membership function is:

$$\mu_{\tilde{X}_{41}}(x) = \begin{cases} (x-5.01)/0.03 & 5.01 \le x \le 5.04 \\ (5.07-x)/0.03 & 5.04 \le x \le 5.07 \end{cases}$$

Its α-cut is: $[(X_{41})^L_\alpha, (X_{41})^U_\alpha]$ = [5.01+0.03α, 5.07–0.03α].

In this example there are 32 fuzzy numbers. Therefore, 32 variables are involved in solving the pair of bound-constrained nonlinear programs associated with each α-cut. By using the solution method devised in the preceding section, the fuzzy MPI of the eight new districts at eleven α-cuts, α= 0, 0.1,…, 1.0, are calculated, with the results shown in Table 3. Figure 3 is a pictorial presentation of the eight fuzzy MPIs.

Table 1. Data of seventeen forests in Taiwan.

Forests	Input: Land (10^3 ha)	Input: Labor (persons)	Input: Expend. ($\$10^6$)	Input: Init. stock (10^6 m^3)	Output: Timber (10^3 m^3)	Output: Soil con. (10^6 m^3)	Output: Recreation (10^3 visits)
1 Wen Shan	60.85	270.0	4.11	5.04	15.85	5.17	14.57
2 Chu Tung	108.46	597.9	9.30	13.45	47.19	18.86	7.00
3 Ta Chia	79.06	421.4	6.35	8.27	21.57	10.48	33.73
4 Tah Sue Shan	59.66	860.1	12.28	10.95	8.41	11.71	9.64
5 Pu Li	84.50	271.0	4.33	9.93	39.04	12.25	0.00
6 Luan Ta	127.28	592.0	10.45	13.36	57.11	13.81	0.00
7 Yu Shan	98.80	863.0	12.15	8.14	42.81	12.43	399.83
8 Nan Nung	123.14	852.0	8.84	10.86	55.20	9.18	7.56
9 Heng Chung	86.37	285.0	5.35	8.62	39.24	6.88	1,081.89
10 Kuan Shan	227.20	216.1	5.87	24.04	44.08	27.28	0.00
11 Yu Li	146.43	205.0	4.08	15.76	37.30	19.30	0.00
12 Mu Kua	173.48	774.9	12.60	23.03	9.63	23.53	41.86
13 Lan Yang	171.11	2,722.7	14.51	17.84	19.73	18.86	84.00
14 Forest Expl. B.	93.65	1,399.0	128.94	17.58	42.11	17.30	0.00
15 Forest Res. Inst.	13.65	350.9	0.91	1.42	19.07	1.58	0.00
16 Taiwan Univ.	33.52	165.0	1.73	0.38	13.57	0.50	1,061.48
17 Chung Hsin Univ.	8.23	49.0	0.30	1.59	3.86	1.57	67.73

Table 2. Data of the eight new districts before and after reorganization.

Districts	Input: Land (10^3 ha)	Input: Labor (persons)	Input: Expend. ($\$10^6$)	Input: Init. stock (10^6 m^3)	Output: Timber (10^3 m^3)	Output: Soil con. (10^6 m^3)	Output: Recreation (10^3 visits)
Before reorganization							
1 Lo Tung	189.36	2,803.5	15.75	18.82	19.73	20.12	84.00
2 Hsin Chu	151.07	787.1	12.18	17.50	58.28	22.79	280.85
3 Tung Shi	138.71	1,281.0	18.63	19.22	29.98	22.18	43.36
4 Nan Tou	211.78	863.0	14.78	23.29	96.15	26.07	0.00
5 Chia Yi	121.20	1,018.0	13.76	10.04	47.76	13.24	399.83
6 Pin Tung	187.10	981.7	12.58	17.44	89.49	15.41	1,238.98
7 Tai Tung	227.20	216.1	5.87	24.04	44.08	27.28	0.00
8 Hua Lien	319.91	979.9	16.67	38.78	46.93	42.83	41.88
After reorganization							
1 Lo Tung	175.73	442.5	11.67	16.04	3.09	16.04	119.46
2 Hsin Chu	162.81	417.9	12.93	26.10	12.45	26.10	287.26
3 Tung Shi	138.41	561.3	20.87	23.48	4.51	23.48	247.53
4 Nan Tou	211.82	462.4	17.30	23.53	11.16	23.53	0.00
5 Chia Yi	139.52	587.1	8.30	13.16	3.52	13.21	845.38
6 Pin Tung	196.05	345.8	12.17	15.88	11.61	15.88	964.04

7 Tai Tung	226.55	202.3	5.91	26.80	15.11	26.80	159.31
8 Hua Lien	320.85	525.9	12.02	44.13	3.72	44.11	61.70

Table 3. Fuzzy MPIs of the eight new forest districts expressed by eleven α-cuts.

Districts		α=0	α=0.1	α=0.2	α=0.3	α=0.4	α=0.5	α=0.6	α=0.7	α=0.8	α=0.9	α=1
1 Lo Tung	L	0.8873,	0.8906,	0.8948,	0.8990,	0.9030,	0.9066,	0.9108,	0.9154,	0.9200,	0.9248,	0.9295
	U	0.9799,	0.9748,	0.9696,	0.9644,	0.9593,	0.9542,	0.9492,	0.9442,	0.9393,	0.9344,	0.9295
2 Hsin Chu	L	0.9816,	0.9828,	0.9839,	0.9851,	0.9862,	0.9874,	0.9886,	0.9897,	0.9909,	0.9921,	0.9933
	U	1.0032,	1.0022,	1.0012,	1.0002,	0.9992,	0.9982,	0.9972,	0.9962,	0.9956,	0.9945,	0.9933
3 Tung Shi	L	1.0753,	1.0770,	1.0787,	1.0804,	1.0821,	1.0838,	1.0855,	1.0873,	1.0890,	1.0907,	1.0927
	U	1.1178,	1.1158,	1.1132,	1.1106,	1.1081,	1.1056,	1.1030,	1.1003,	1.0979,	1.0953,	1.0927
4 Nan Tou	L	0.6345,	0.6363,	0.6381,	0.6401,	0.6424,	0.6448,	0.6471,	0.6495,	0.6519,	0.6543,	0.6567
	U	0.6811,	0.6786,	0.6762,	0.6737,	0.6712,	0.6688,	0.6663,	0.6639,	0.6615,	0.6591,	0.6567
5 Chia Yi	L	0.9418,	0.9455,	0.9490,	0.9525,	0.9560,	0.9596,	0.9631,	0.9668,	0.9705,	0.9742,	0.9780
	U	1.0089,	1.0061,	1.0033,	1.0005,	0.9978,	0.9950,	0.9923,	0.9892,	0.9854,	0.9817,	0.9780
6 Pin Tung	L	0.7783,	0.7787,	0.7790,	0.7793,	0.7797,	0.7800,	0.7804,	0.7807,	0.7811,	0.7814,	0.7818
	U	0.7853,	0.7850,	0.7846,	0.7843,	0.7839,	0.7836,	0.7832,	0.7828,	0.7825,	0.7821,	0.7818
7 Tai Tung	L	1.0652,	1.0672,	1.0691,	1.0709,	1.0728,	1.0747,	1.0766,	1.0785,	1.0804,	1.0823,	1.0843
	U	1.0941,	1.0931,	1.0922,	1.0912,	1.0902,	1.0892,	1.0882,	1.0872,	1.0862,	1.0853,	1.0843
8 Hua Lien	L	0.9739,	0.9751,	0.9762,	0.9774,	0.9785,	0.9797,	0.9809,	0.9821,	0.9833,	0.9844,	0.9856
	U	0.9975,	0.9963,	0.9952,	0.9940,	0.9928,	0.9916,	0.9904,	0.9892,	0.9880,	0.9868,	0.9856

L: lower bound, U: upper bound

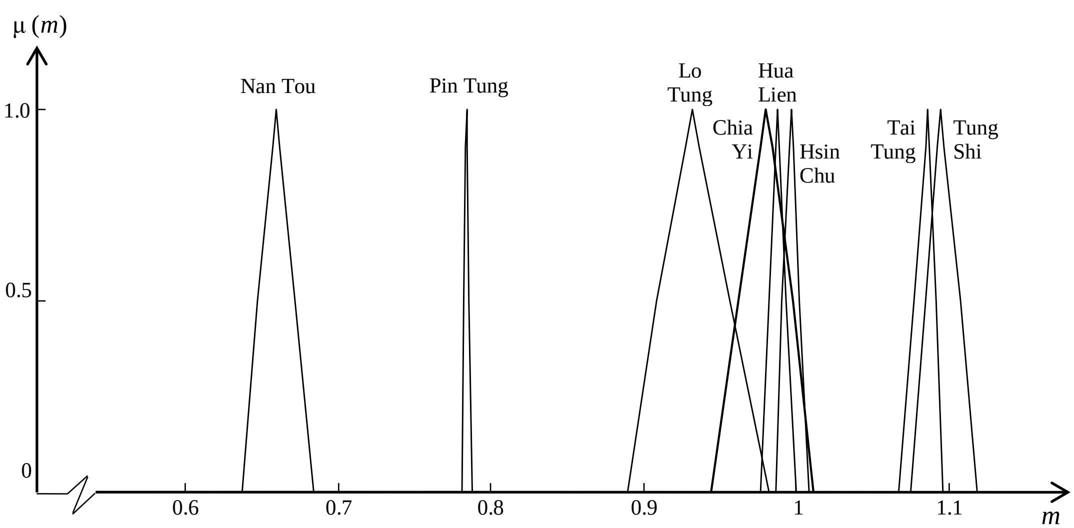

Figure 3. Fuzzy MPIs of the eight forest districts of the numerical example.

The α=1 cuts, with only one value, are the same as the MPIs calculated from precise observations. Only two districts, Tung Shi and Tai Tung, have an MPI of greater than 1, indicating that their efficiencies improved after the reorganization. The order of improvement of the eight districts is Tung Shi, Tai Tung, Hsin Chu, Hua Lien, Chia Yi, Lo Tung, Pin Tung, and Nan Tou. The least precise situation of the α=0 cuts indicate that Hsin Chu and Chia Yi also have possibilities of having improved after the reorganization, since their upper-bound values are greater than 1.

The α=0 cuts also indicate that Tung Shi and Tai Tung are clearly better than the other districts, as their α=0 cuts do not overlap with those of others. The lower bounds of greater than 1 also indicate that their performances have definitely improved. Tung Shi, in general, performs better than Tai Tung; however, there is still possibility that the latter will outperform the former. Hsin Chu and Hua Lien have a large portion of their α=0 cuts overlapping each other, although the former seems to be performing better. Visually, Chia Yi is outperformed by Hua Lien; yet interestingly, the former has some possibility of being evaluated as having improved (with an MPI greater than 1), while the latter is evaluated as definitely worsening. Lo Tung performs worse than Chia Yi, although it has some possibility of having performed better than the latter. Ping Tung and Nan Tou are clearly worse than the others, as their α=0 cuts trail those of the others. The narrow range of the α=0 cut of Pin Tung indicates that its MPI is reliably represented by its most likely value of 0.7818, while the wide range of the α=0 cut of Lo Tung indicates that its most likely value of 0.9295 is not representative for its MPI.

This example shows that when the imprecise data is assumed to be precise, only two districts have their performance improved, and the order of performance improvement of the eight districts is clearly determined. When the imprecise data is represented by fuzzy numbers, the results show that two more districts may be evaluated as having an improved performance. The order of performance improvement of the eight districts is thus no longer certain. Tung Shi and Tai Tung are better than the others, while either has the possibility of outperforming the other. Hsin Chu and Hua Lien are definitely better than Lo Tung, Nan To, and Pin Tung, and similarly either has the possibility of outperforming the other. Chia Yi is likely to be worse than Hsin Chu and Hua Lien, although it has some possibility of outperforming them. Pin Tung and Nan Tou, in sequence, are definitely worse than the others, just as in the precise case.

6 CONCLUSION

The Malmquist productivity index (MPI) is an effective measure for the change in performance of a unit between two periods. The conventional MPI is devised for deterministic cases. When some of the observations have imprecise values or qualitative measures, and can be expressed as fuzzy numbers, this paper demonstrates that the corresponding MPIs also are fuzzy numbers.

By applying the extension principle, this paper is able to formulate the problem as a pair of two-level mathematical programs, one for each bound of the α-cut of the fuzzy MPI. The two-level mathematical program can be converted to a bound constrained nonlinear program, and a modified quasi-Newton method is devised for a solution. The case of the reorganization of Taiwanese forests, where the forest stocks cannot be measured precisely and are represented by fuzzy numbers, illustrates the validity of the model and the solution method devised in this paper.

The results of the reorganization problem show that the fuzzy approach provides more information than that acquired by assuming the imprecise observations to be precise. Specifically, Hsin Chu and Chia Yi Districts, which are considered as having a worsening performance, have some possibility of being evaluated as having improved. Several districts, which are considered as worse than some others, have the possibility of outperforming them. Fuzzy measures thus alert decision makers not to be too confident with some results, when they would be confident if the imprecise observations were assumed to have precise values.

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