Double Domination Number and Connectivity of Graphs

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ABSTRACT

In a graph G, a vertex dominates itself and its neighbours. A subset S of V is called a dominating set in G if every vertex in V is dominated by at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A set $S \subseteq V$ is called a double dominating set of a graph G if every vertex in V is dominated by at least two vertices in S. The minimum cardinality of a double dominating set is called double domination number of G and is denoted by dd(G). The connectivity $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper we find an upper bound for the sum of the double domination number and connectivity of a graph and characterize the corresponding extremal graphs.

Key Words: Domination, Domination number, Double domination, double domination number and Connectivity.

1. INTRODUCTION

The graph G = (V,E) we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by m and n respectively. The degree of any vertex u in G is the number of edges incident with u and is denoted by d(u). The minimum and maximum degree of a graph G is denoted by $\delta(G)$ *and* $\Delta(G)$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et .al [2, 3].

Let $v \in V$. The open neighbourhood and closed neighbourhood of v are denoted by N(v) and $N[v] = N(v) \cup \{v\}$ respectively.If $S \subseteq V$ then $N(S) = \underset{v \in s}{\cup} N(v)$ for all $v \in S$ *and* $N[s] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbour set of u with respect to S is defined by $pn[u,S] = \{v : N[v] \cap S = \{u\}\}$. H($m_1$,$m_2$,....$m_n$) denotes the graph obtained from the graph H by pasting m_i edges to the vertex $v_i \in V(H), 1 \le i \le n$ H(P_{m1},P_{m2},.......P_{mn}) is the graph obtained from the graph H by attaching the end vertex of P_{mi} to the vertex v_i, in H , $1 \le i \le n$. Bistar B(r,s) is a graph obtained from $K_{1,r}$ *and* $K_{1,s}$ by joining its centre vertices by an edge.

In a graph G, a vertex dominates itself and its neighbours. A subset S of V is called a dominating set in G if every vertex in V is dominated by at least one vertex in

S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. Harary and Haynes [4] introduced the concept of double domination in graphs. A set $S \subseteq V$ is called a double dominating set of a graph G if every vertex in V is dominated by atleast two vertices in S. The minimum cardinality of double dominating set is called double domination number of G and is denoted by dd(G). The connectivity $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. J.Paulraj Joseph and S.Arumugam [5] proved that $\gamma(G)+\kappa(G) \le n$ and characterized the corresponding extremal graphs. In this paper, we obtain an upper bound for the sum of the double domination number and connectivity of a graph and characterize the corresponding extremal graphs. We use the following theorems.

Theorem 1.1. [2] For any graph G, dd(G)≤n.

Theorem 1.2. [1] For a graph G, $\kappa(G) \le \delta(G)$.

2. MAIN RESULTS

Theorem 2.1. For any connected graph G $dd(G)+\kappa(G) \le 2n-1$ and equality holds if and only if G is isomorphic to K_2.

Proof.
$dd(G)+\kappa(G) \le n+\delta \le n+n-1 = 2n-1$.
Let $dd(G)+\kappa(G) = 2n-1$. Then dd(G) = n and $\kappa(G) = n-1$. Then G is a complete graph on n vertices. Since $dd(K_n)$ = 2 we have n = 2 . Hence G is isomorphic to K_2. The converse is obvious. □

Theorem 2.2. For any connected graph G, $dd(G)+\kappa(G) = 2n-2$ if and only if G is isomorphic to K_3 or $K_{1,2}$.

Proof. Let $.\, dd(G)+\kappa(G) = 2n-2.$ Then there are two cases to consider.

(i) $dd(G) = n-1$ *and* $\kappa(G) = n-1$

(ii) $dd(G) = n$ *and* $\kappa(G) = n-2$

Case 1. $dd(G) = n-1$ *and* $\kappa(G) = n-1$

Then G is a complete graph on n vertices. Since $dd(K_n)$= 2 we have n = 3 . Hence G is isomorphic to K_3.

Case 2 . $dd(G) = n$ *and* $\kappa(G) = n-2$

Then $n-2 \le \delta(G)$. If $\delta = n-1$ then G is a complete graph which is a contradiction. Hence $\delta = n-2$. Then G is isomorphic to K_n-Y where Y is a matching in K_n . Then dd(G) = 2 or 3. Since dd(G) =2 is isomorphic we have n = 3. Hence G is isomorphic to $K_{1,2}$. The converse is obvious. □

Theorem 2.3. For any connected graph G $dd(G)+\kappa(G) = 2n-3$ if and only if G is isomorphic to K_4 or C_4 or P_4 or $K_{1,3}$.

Proof. Let $dd(G)+\kappa(G)=2n-3$. Then there are three cases to consider

(i) $dd(G)=n-2$ *and* $\kappa(G)=n-1$

(ii) $dd(G)=n-1$ *and* $\kappa(G)=n-2$

(iii) $dd(G)=n$ *and* $\kappa(G)=n-3$

Case 1. $dd(G)=n-2$ *and* $\kappa(G)=n-1$

Then G is a complete graph on n vertices. Since dd(K_n)=2 we have n = 4 . Hence G is isomorphic to K_4.

Case 2. $dd(G)=n-1$ *and* $\kappa(G)=n-2$

Then $n-2\le\delta(G)$. *If* $\delta=n-1$ then G is a complete graph which gives a contradiction. Hence $\delta(G)=n-2$. Then G is isomorphic to K_n - Y where Y is a matching in K_n. Then dd(G) = 2 or 3. If dd(G) = 2 then n = 3. Hence G is isomorphic to $K_{1,2}$ which is a contradiction. Thus dd(G) = 3. Then n = 4 and hence G is isomorphic to C_4.

Case 3. $dd(G)=n$ *and* $\kappa(G)=n-3$

Then n -3 $\le\delta$. *If* $\delta=n-1$ then G is a complete graph which is a contradiction. If $\delta=n-2$ then G is isomorphic to K_n-Y where Y is a matching in K_n. Then dd(G) =2 or 3. Then n =2 or 3 which is a contradiction to $\kappa(G)=n-3$. Hence $\delta=n-3$. Let X be vertex cut of G with $|X|=n-3$ *and let* $V-X=\{x_1,x_2,x_3\}$, $X=\{v_1,v_2,v_3.....,v_{n-3}\}$.

Sub case 3.1. $\langle V-X\rangle=\overline{K_3}$

Then every vertex of V-X is adjacent to all the vertices of X. Then $(V-X)\cup\{v_1\}$ *is a* double dominating set of G hence dd (G) ≤ 4. This gives n≤4. Since n≤3 is impossible, we have n = 4. Hence G is isomorphic to $K_{1,3}$.

Sub case 3.2. $\langle V-X\rangle=K_1\cup K_2$

Let $x_1x_2\in E(G)$. Then x_3 is adjacent to all the vertices in X and x_1, x_2 are not adjacent to at most one vertex in X. If $v_1\notin N(x_1)\cup N(x_2)$ then $(V-X)\cup\{v_1\}$ is a double dominating set of G and hence dd(G)≤4. This gives n = 4 which is a contradiction to G is a connected graph. So all $v_i\in$ either N(x_1) or N(x_2) or both. . Then $(V-X)\cup\{v_i\}$ *is* a double dominating set of G. Hence dd(G)≤4 and then n = 4 . Then G is isomorphic to P_4 or $K_3(1,0,0)$. But dd[$K_3(1,0,0)$]=3≠n which is a contradiction . The converse is obvious. □

Theorem 2.4. For any connected graph G $dd(G)+\kappa(G)=2n-4$ if and only if G is isomorphic to K_4 – e or $K_{1,4}$ or K_3 (1,0,0) or B(1,2) or K_5 – Y where Y is a matching on K_5 with $|Y|=2$

Proof. Let $dd(G)+\kappa(G)=2n-4$. Then there are four cases to consider

(i) $dd(G)=n-3$ *and* $\kappa(G)=n-1$

(ii) $dd(G)=n-2$ *and* $\kappa(G)=n-2$

(iii) $dd(G)=n-1$ *and* $\kappa(G)=n-3$

(iv) $dd(G)=n$ *and* $\kappa(G)=n-4$

Case 1. $dd(G)=n-3$ *and* $\kappa(G)=n-1$

Then G is a complete graph on n vertices. Since dd(G) = 2 we have n = 5. Hence G is isomorphic to K_5.

Case 2. $dd(G)=n-2$ *and* $\kappa(G)=n-2$

Then $n-2 \le \delta$. $If\ \delta = n-1$ then G is a complete graph which is a contradiction. If $\delta(G) = n-2$. Then G is isomorphic to K_n - Y is a matching in K_n. Then dd(G)=2 or 3 . If dd(G) =2, then n = 4 then G is either C_4 or K_4- e. But dd(C_4)=3≠n-2. Hence G is isomorphic to K_4-e. If dd(G) = 3 then n = 5 and hence G is isomorphic to K_5- Y where Y is a matching on K_5 with $|Y| = 2$.

Case 3. $dd(G) = n-1\ and\ \kappa(G) = n-3$

Then $n-3 \le \delta$. $If\ \delta = n-1$ then G is a complete graph which is a contradiction. If $\delta = n-2$ then G is isomorphic to K_n- Y where Y is a matching in K_n. Then dd(G)=2 or 3. Then n = 3 or 4. Since n = 3 is impossible, we have n = 4. Then G is either K_4 – e or C_4. For these two graphs $\kappa(G) \ne n-3$ which is a contradiction. Hence $\delta = n-3$.

Let X be the vertex cut of G

with $|X| = n-3$ and let V-X = {x_1,x_2,x_3},

X = {v_1, v_2,....,v_{n-3}}.

Sub Case 3.1. $\langle V-X \rangle = \overline{K_3}$

Then every vertex of V–X is adjacent to all the vertices of X . Then $(V-X) \cup \{v_1\}$ is a double dominating set of G and hence dd(G) ≤ 4. This gives n≤5. Since n≤3 is impossible we have n = 4 or 5. If n = 4 then G is isomorphic to $K_{1,3}$ which is a contradiction. If n = 5 then the graph G has dd(G) = 2 or 3 which is a contradiction.

Subcase 3.2. $\langle V-X \rangle = K_1 \cup K_2$

Let $x_1x_2 \in E(G)$. Then x_3 is adjacent to all the vertices in X and x_1, x_2 are not adjacent to at most one vertex in X. If $v_1 \notin N(x_1) \cup N(x_2)$ then $(V-X) \cup \{v_1\}$ is a double dominating set of G and hence dd(G)≤4. This gives n = 5 . For this graph $\kappa(G) = 1$ which is a contradiction. So all v_i, either $v_i \in N(x_1)\ or\ v_i \in N(x_2)$ or both. Then $(V-X) \cup \{v_i\}$ is a double dominating set of G. Hence dd(G)≤4 and then n = 4 or 5. If n = 4 then G is isomorphic to C_5 or $C_3(P_3,P_1,P_1)$. But $\kappa[c_3(P_3, P_1, P_1)] = 1 \ne n-3.$ Hence G is isomorphic to C_5.

Case 4. $dd(G) = n\ and\ \kappa(G) = n-4$

Then $n-4 \le \delta(G)$. $If\ \delta = n-1$ then G is a complete graph which is a contradiction. $If\ \delta = n-2$ then G is isomorphic to K_n – Y where Y is a matching in K_n. Then dd(G)=2 or 3. Then n = 2 or 3 which is a contradiction to $\kappa(G) = n-4$. Suppose $\delta(G) = n-3$. Let X be the vertex cut of G with $|X| = n-4\ and\ let\ X = \{v_1, v_2, ...v_{n-4}\}$

$V-X = \{x_1, x_2, x_3, x_4\}$. If $\langle V-X \rangle$ contains an isolated vertex then $\delta(G) \le n-4$ which is contradiction. Hence $\langle V-X \rangle$ is isomorphic to $K_2 \cup K_2$. Also every vertex of V – X is adjacent to all the vertices of X. $Let\ x_1x_2, x_3x_4 \in E(G)$. Then {$x_1$,$x_3$,$v_1$} is a double dominating set of G. Then dd(G)≤3. Hence n≤3 which is contradiction. Thus $\delta(G) = n-4$.

Subcase 4.1. $\langle V-X\rangle=\overline{K_4}$

Then every vertex of V – X is adjacent to all the vertices in X. Suppose $E(\langle X\rangle)=\phi.$ Then $|X|\leq 4$ and hence G is isomorphic to $K_{s,4}$ where s = 1,2,3,4. If s ≠1 then dd(G)=3 or 4 which is a contradiction to dd(G) = n. Hence G is isomorphic to $K_{1,4}$. Suppose $E(\langle X\rangle)=\phi.$ If any one of the vertex in X say v_i is adjacent to all the vertices in X and hence dd(G)≤3 which gives n≤3 which is a contradiction. Hence every vertex in X is not adjacent to at least one vertex in X . Then $\{x_1,x_2.v_1.v_2\}$is a double dominating set of G and hence dd(G)≤4. Then n≤4 which is a contradiction to $\kappa(G)=n-4$.

Subcase 4.2. $\langle V-X\rangle=P_3\cup K_1$

Let x_1 be the isolated vertex in $\langle V-X\rangle$ and (x_2, x_3, x_4) be a path. Then x_1 is adjacent to all the vertices in X and x_2,x_4 are not adjacent to at most one vetex in X and hence $\{x_1,x_2.x_4,v_1.v_2\}$where $v_1\in N(x_1)\cap X$ *and* $v_2\in N(x_2)\cap X$ is a double dominating set of G and hence dd(G)≤5. Thus n = 5. Then G is isomorphic to P_5 or $C_4(1,0,0,0)$ or $K_3(1,1,0)$ or $(K_4$- e)(1,0,0,0) . All these graph dd(G)≠n which is contradiction.

Subcase 4.3. $\langle V-X\rangle=K_3\cup K_1$

Let x_1 be the isolated vertex in $\langle V-X\rangle$ and $\langle\{x_2,x_3,x_4\}\rangle$ be a complete graph. Then x_1 is adjacent to all the vertex in X and x_2,x_3,x_4 are not adjacent to at most two vertices in X and hence $\{x_1,x_2,x_3,v_1,v_2\}$ where $v_1,v_2\in X-N(x_2\cup x_3)$ is a double dominating set of G and hence n = 5 . All these graph dd(G)≠n.

Subcase 4.4. $\langle V-X\rangle=K_2\cup\overline{K_2}$

Let $x_1x_2\in E(G)$ and $x_3x_4\in E(\overline{G})$. Then each x_i, i = 1 or 2 is non adjacent to atmost one vertex in X and each x_j, j = 3 or 4 is adjacent to all the verticesin X. Then $\{x_1,x_2,x_3,x_4,v_1\}$is a double dominating set of G and hence dd(G)≤5. Then n≤5. Since $\kappa(G)=n-4$ we have n = 5. Then G is isomorphic to either B(1,2) or $K_3(2,0,0)$. Since dd[$K_3(2,0,0)$]≠5 G is isomorphic to B(1,2).

Subcase 4.5. $\langle V-X\rangle=K_2\cup K_2$

Let $x_1x_2,\,x_3x_4\in E(G).$ Since $\delta(G)=n-4,$ each x_i is non adjacent to atmost one vertex in X. Then at most one vertex say $v_1\in X$ such that $|N(v_1)\cap(V-X)|=1$. If all $v_i\in X$ such that $|N(v_i)\cap(V-X)|\geq 2$ then $\{x_1,x_2,x_3,x_4\}$is a double dominating set of G and hence n = 4 which is a contradiction. Then $|N(v_1)\cap(V-X)|=1$ and $|N(v_i)\cap(V-X)|\geq 2, i\neq 1.$ Then $\{v_1,x_1,x_2,x_3,x_4\}$ is a double dominating set of G and hence n = 5 which gives a contradiction to $|N(v_1)\cap(V-X)|=1$ and $|N(v_i)\cap(V-X)|\geq 2.$ The converse is obvious.

3. Conclusion

In this paper we found an upper bound for the sum of double domination number and connectivity of graphs and characterized the corresponding extremal graphs. Similarly double domination number with other graph theoretical parameters can be considered.

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